

# Modeling and Simulations of Liquid for Precise Level Measurements

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## Abstract

Precise liquid level measurement is one of the critical research area. A variety of level sensors are available in the market for different applications. They are classified into point, continuous, contact, non-contact, liquid, solid, electromagnetic or electromechanical etc. These sensors show inaccuracy in liquid level measurement and that result due to several dynamic environmental factors like sloshing, temperature variations, noise, salinity, density, dust, dielectric medium constant, pressure, mechanical shock, tank size and shape. All of above mentioned factors limit the performance of level sensors in any vibrant environment. We have addressed sloshing of liquid factor in our research work. Sloshing phenomena refers to the dynamic movement of any liquid inside a container. It is the result of all kinds of external excitation.

First, we have formulated a Boundary Value Problem (BVP) of liquid in a specified container whose height, width, length is 10 m, 6 m and 10 m respectively for particularly our work. The final solution is closed form expression of mathematical model of surface wave motion that leads us to the velocity potential. It is implemented in MATLAB for three different waves that are dispersive wave, circular wave, and sinusoidal wave. We have analyzed the resonance frequencies of sloshing of these three types of waves that lies in range 0.5-7 Hz. Measurement of these resonant frequencies helps us to minimize the effect of sloshing in a liquid by using a low pass filter.

## 1. Introduction

Reasonable and dependent level measuring technology is of great importance for industrial, domestic and various other applications such as fuel storage, flood warning, remote monitoring, sea level monitoring, tsunami warning, biochemical industry and water surface level control. The need of precise and accurate level measuring system is increased with demand of advanced automated processing systems. Both accuracy and precision is difficult to achieve in a level measurement system. For example improvement of sample size in a faulty system generally increases precision [1-2].

Level sensors are the devices that measure the level of the free-flowing substances. Selection of an appropriate type of level sensor that suits to the application requirement is very important.

A variety of level sensors are available in the market for different applications. They are classified into following categories;

- Point or Continuous
- Contacting or Non-Contacting
- Liquids or Solids
- Electromagnetic or Electromechanical

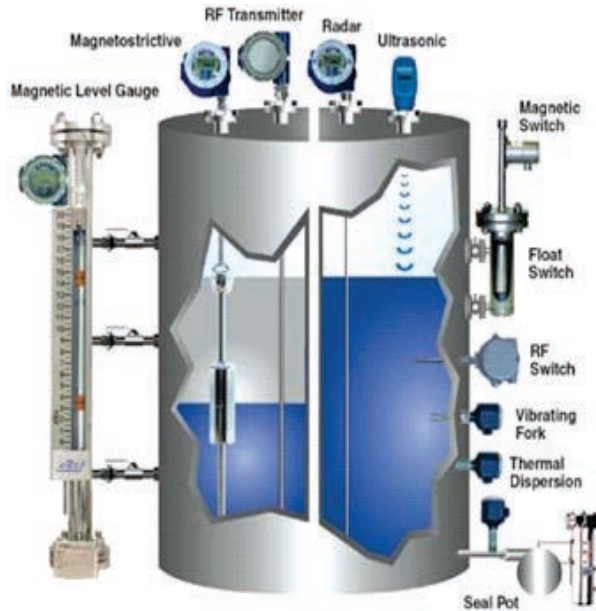


Fig. 1. Different Types of Level Sensors [3]

Point level measurement sensors are used as a marker to trigger low or high level alarm in tanks through liquid level sensing. To get most refined liquid level observation of an entire system, continuous level sensors are used. Contact level sensors are the ones that are directly in contact with the fluid i.e. touching the fluid. Whereas the non-contact level sensors are those that are not directly touching the fluid instead they are attached to the container [4].

The correct selection of a sensor and then the subsequent correct application of that sensor will make the level measurement a successful one. However, no sensor is perfect, each sensor shows limited behavior in dynamic environment due to many aspects such as sloshing effect of liquid, noise, temperature variations and other contaminations that cause inaccuracy in sensor's reading [5-7].

Surface waves propagate along the liquid edge and interface between liquid and air. Surface waves have been the subject of research for decades, however practical implementation with the help of mathematical modeling is limited and there are some research factors related to

surface waves such sloshing, contamination, salinity, density and significance of turbulent motion. So, there is a need of development of robust level measurement system which will be able to consider all these environmental factors and give us accurate reading regardless of the level sensor limitations. In this paper, sloshing factor of liquid waves with the help of mathematical modeling is addressed.

Motion-induced sloshing has been a conventional control problem. It was first met in the control of guided missiles. The dynamic forces resulting from the motion of these fluid-filled missiles can be considerable and result in instabilities. The first analytical solution to this a problem was addressed by Westergard [8]. Aerospace vehicles, maritime and transportation applications really need accurate evaluation of the sloshing because irregular motion of liquid fuel can cause instability and structural damage [9-10].

Real time observation of sloshing effect helps to establish integrity of mathematical model of surface wave motion. It involves the use of mathematics as the core of dynamic approach to infer quantitative information about the movement of fluid due to sloshing. To achieve this objective a boundary value problem is derived to model surface wave motion in floating tank. Mathematical modeling of the surface wave motion will give us a deeper insight in the effect of sloshing on the liquids in dynamic state. Mathematical modeling approach is most likely to be successful for accomplishment of this objective because it results in analytical expressions which can be used for prediction of sloshing in to the future according to boundary values.

## II. Block Diagram

Figure 2 shows the block diagram of complete process for liquid level sensing. In this paper, post-acquisition process of liquid level sensing is presented that is consists of mathematical model and simulation of water waves. First it takes the dimensions of the container than calculates the boundary values for liquid in the container which leads to the approximate solution.

## III. Mathematical Modeling

### A. Finding a Governing Equation

The variables that are used to develop the model in the tank are shown in Figure 3. Z-axis shows the vertical direction, y-axis points into the page, 'h' represents depth of the fluid and x represents the horizontal direction.

$z = 0$  represents the undisturbed water level and the unknown free surface wave motion is described by a function  $z = \eta(x, y, t)$ . When  $z = \eta$  the water is moving in the form of curvature or making any wave. Fluid depth is constant and finite below the undisturbed water level which is about  $z = 0$ . Here assumption is made that fluid

lies below an infinite layer of still air.

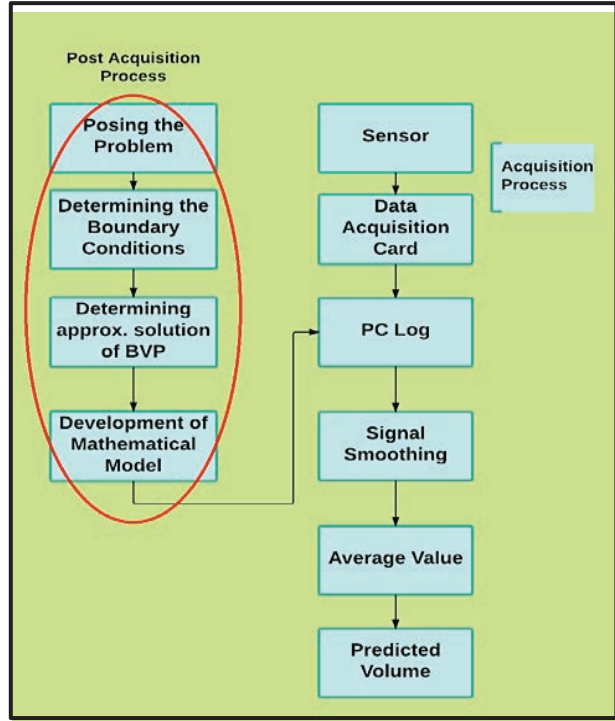


Fig. 2. Liquid Level Sensing through Mathematical Modeling

The movement of every point inside the liquid is described by the velocity field  $u(x, y, z, t) = (u_1, u_2, u_3)$ , where  $u_1, u_2, u_3$  are functions of  $x, y, z$  and  $t$  and  $x, y, z, t$  are partial differentiation with respect to these variables. Leibnitz forms such as  $\partial\phi/\partial t$  are used for emphasis.

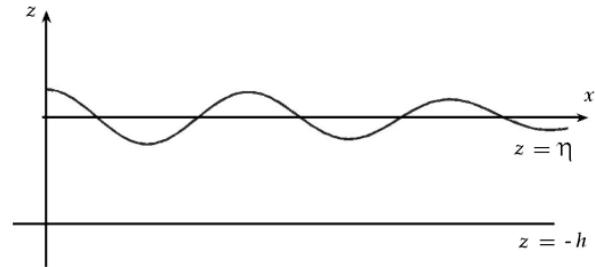


Fig. 3. Variables Indicating Context

### A. Assumptions

Here following three assumptions for developing mathematical model of liquid waves are considered to make derivations easy and simplified. These assumptions are

- The fluid has no viscosity
- $\nabla \times u = 0$  (expressed in terms of velocity field)
- $\nabla \cdot u = 0$  (vector field is incompressible) (1)

Where  $\nabla = (\partial x, \partial y, \partial z)$

According to standard theorem, if  $\nabla \times u = 0$  then  $\phi(x, y, z, t)$  is a scalar function such that  $u = \nabla\phi$ . When we combine

this definition of  $\phi$  with Eq.1 then it yields  $\nabla \cdot \nabla \phi = 0$ , which shows its second and third assumption result in Laplace equation

$$\phi = 0 \quad (2)$$

Above equation means that  $\phi_{xx} + \phi_{yy} + \phi_{zz} = 0$

The unknown surface behavior can be found by determining the velocity field  $u$ . With the help of second order partial differential equation for  $\phi$ ,  $u$  can be constructed. The solution of  $\phi$  depends upon boundary conditions that explain this surface wave motion problem.

### C. Determining Boundary Conditions

To derive boundary value problem, following conditions are assume that all motion in the problem is uniform and in the y-direction and

- Velocity field  $u(x, z, t) = (u_1, u_3)$
- Velocity potential  $\phi(x, z, t)$  which satisfies  $\phi_{xx} + \phi_{zz} = 0$
- Free Surface  $z = \eta(x, t)$

The water particle velocity under linear waves is maximum at the surface and decrease in magnitude with depth. Vertical component of velocity of water at bottom surface is considered zero when  $z = -h$  which can be represented mathematically by the following equation;

$$u = (u_1, u_3) \quad (3)$$

$$u_3 \Big|_{z=-h} = 0 \quad (4)$$

$$\text{As } u = \nabla \phi = (\phi_x, \phi_z) \quad (5)$$

Now the above boundary condition becomes

$$\frac{\partial \phi}{\partial z} \Big|_{z=-h} = 0 \quad (6)$$

According to dynamic boundary condition, when  $z = \eta(x, t)$  there is pressure jump across the liquid surface and this pressure jump depends on the surface tension of liquid. Pressure jump across the liquid surface and surface tension are inversely proportional to each other.

As mention in Eq.1,  $\nabla \cdot u = 0$ , we begin with Euler equation of motion for irrotational flow of velocity and we have

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} (\nabla \phi) = -\nabla \left( \frac{p}{\rho} + \frac{1}{2} u^2 + g z \right) \quad (7)$$

Here

Flow of velocity  $u = \nabla \phi$

Pressure of fluid  $= p$

Density of fluid  $= \rho$

Influence of gravity  $= g$

Where  $u^2 = u \cdot u$  that is a standard in fluid dynamics and

this form of Euler equation is derived from Newton's Second Law of Motion.

$$\frac{\partial \phi}{\partial t} + \left( \frac{p}{\rho} + \frac{1}{2} u^2 + g z \right) = f(t) \quad (8)$$

This is called Bernoulli's equation for unsteady irrotational flow of fluid. If we absorb the function  $f(t)$  into velocity potential  $\phi$ , the equation can be re-written as the following equation.

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2} u^2 + g z = 0 \quad (9)$$

The reconstruction of  $u = \nabla \phi$  is independent of  $f(t)$ . Under the assumption of small velocity, this equation can be linearized (i.e., the very small term  $u^2$  may be neglected) to the following equation.

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + g z = 0 \quad (10)$$

Motion of free surface is represented by  $z = \eta$ . The pressure difference across this surface is represented by the following equation.

$$P = P - P_a \quad (11)$$

When  $z = \eta$ , pressure difference across the liquid surface is equal to difference of pressure immediately below the liquid surface and atmospheric pressure immediately above the surface.

Air above liquid is stable and pressure exerted by still air on liquid surface is constant. So we can assume that which eliminates the effect of atmospheric pressure from equation. As the pressure difference is balanced by a force called surface tension force. Surface tension acts tangentially towards the water surface. This can be explained mathematically by the following equation.

$$p = p - p_a = -\sigma K \quad (12)$$

$$p = p = -\sigma K \quad (13)$$

Where,  $K$  = curvature of the surface.

When the air above the water is stable then there is no pressure on liquid. So there is no effect on surface tension of liquid and liquid is therefore still. Surface tension of liquid is inversely proportional to the curvature of surface. As long as the liquid retains its flat plan with the help of surface tension there will be no curvature.

Now put the value of pressure difference which is  $p = -\sigma K$  in linearized Bernoulli equation which is Eq.10

$$\frac{\partial \phi}{\partial t} - \frac{\sigma}{\rho} K + g \eta = 0 \quad (z = \eta) \quad (14)$$

$$K = \frac{\eta_{xx}}{[1 + \eta_x^2]^{3/2}} \approx \eta_{xx} \quad (15)$$

Scaling and non-dimensionalization arguments used for

the approximation of K are essential tools in developing mathematical models. In this case,  $\eta$  is a non-linear function [11-12]. Now put the value of K in Eq.14

$$\frac{\partial \phi}{\partial t} + g\eta - \frac{\sigma}{\rho} \eta_{xx} = 0 \quad (z = \eta) \quad (16)$$

Approximate  $\partial\phi/\partial t|_{z=\eta}$  by the first term in its Taylor series centered at 0.

$$\frac{\partial \phi}{\partial t}\bigg|_{z=\eta} = \frac{\partial \phi}{\partial t}\bigg|_{z=0} + \frac{\partial}{\partial z} \frac{\partial \phi}{\partial t}\bigg|_{z=0} \eta + \dots \approx \frac{\partial \phi}{\partial t}\bigg|_{z=0} \quad (17)$$

Substitute this  $\partial\phi/\partial t|_{z=\eta}$  in Eq.16 and it gives us the dynamic boundary condition.

$$\frac{\partial \phi}{\partial t} + g\eta - \frac{\sigma}{\rho} \eta_{xx} = 0 \quad (18)$$

This can be applied at the known surface which is given by  $z=0$ .

#### D. Kinematic Boundary Condition

According to kinematic boundary condition velocity of fluid at a point must be identical to each point of the surface. When  $\eta_x$  and  $\eta$  are not both small, the motion of surface should match the component of the fluid velocity that is perpendicular to the surface of fluid.

- The velocity of the surface at  $(x, t)$  is
- The velocity of the liquid at the surface is  $\mathbf{u}(x, \eta, t)$

To agree these velocities in the vertical direction requires that  $\partial\eta/\partial t = u_3|_{z=\eta}$ . In terms of the potential function  $\phi$ , this condition is

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z}\bigg|_{z=\eta} \quad (19)$$

Again using the first term in the Taylor Series for  $\frac{\partial \phi}{\partial z}\big|_{z=\eta}$  to approximate with  $\frac{\partial \phi}{\partial z}\big|_{z=0}$  gives

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z}\bigg|_{z=0} \quad (20)$$

#### E. The Boundary Value Problem

Upto to this point boundary value problem for the velocity potential  $\phi$  is formulated. However, there is one major difficulty is that this solution depends upon  $\eta$  which is also unknown till now. Once we find out the value of  $\eta$ , we can lead to the approximate solution for  $\phi$

$$\Delta\phi = 0; \quad (21)$$

$$\frac{\partial \phi}{\partial z} = 0 \quad (z = -h) \quad (22)$$

$$\frac{\partial \phi}{\partial t} = \frac{\sigma}{\rho} \eta_{xx} - g\eta \quad (z = 0) \quad (23)$$

$$\frac{\partial \phi}{\partial t} = \frac{\partial \eta}{\partial t} \quad (z = 0) \quad (24)$$

#### F. Approximate Solution of Boundary Value Problem

##### • Choosing $\eta$

Wave equation solution can be given in the form of even and odd dimensions. The shape of the free space  $\eta$  can be

given by the following equation;

$$\eta(x, t) = a \cos(kx - \omega t) = a \cos(k(x - ct)) \quad (25)$$

Where

$$\text{Wave number} = k = \frac{2\pi}{\lambda}$$

$$\text{Wave speed} = c = \frac{\omega}{k}$$

Angular or circular frequency of travelling cosine wave =  $\omega$

##### • The Velocity Potential

Here  $\phi(x, z, t) = F(x, t)Z(z)$  Boundary condition in Eq.24 requires  $\phi_z = \eta_t$  when  $z=0$

Thus

$$F(x, t)Z'(z) = [a \cos(kx - \omega t)]_t \quad (z = 0) \quad (26)$$

so,

$$F(x, t)Z'(0) = a\omega \sin(kx - \omega t) \quad (27)$$

Now substitute following Eq.28 into the governing partial differential equation and we will get Eq.29

$$\phi(x, z, t) = \sin(kx - \omega t)Z(z) \quad (28)$$

$$\phi_{xx} + \phi_{zz} = 0 \quad (29)$$

With the help of above equations, we found out following equation

$$-k^2 \sin(kx - \omega t)Z(z) + \sin(kx - \omega t)Z''(z) = 0, \quad (30)$$

Eq.30 leads to a differential equation in  $z$

$$Z'(z) = Ake^{kz} - Bke^{-kz} \quad (31)$$

According to boundary condition stated in Eq.22, put the value  $Z'(-h) = 0$  in Eq.31

$$Z'(-h) = Ake^{-kh} - Bke^{kh} = 0 \quad (32)$$

$$Ake^{-kh} - Bke^{kh} = 0$$

$$Z(z) = B(e^{-kz} + e^{2kh}e^{kz}) \quad (33)$$

Now apply kinematic boundary condition mention in Eq.24 and the definition of  $h$  according to Eq.25

$$\phi(x, z, t) = \sin(kx - \omega t)Z(z) \quad (34)$$

$$\omega a \sin(kx - \omega t) = \sin(kx - \omega t)[-kB(1 - e^{2kh})] \quad (35)$$

$$B = \frac{\omega a}{k(e^{2kh} - 1)} \quad (36)$$

$$Z(z) = B(e^{-kz} + e^{2kh}e^{kz}) \quad (37)$$

$$Z(z) = \frac{\omega a}{K(e^{2kh} - 1)}(e^{-kz} + e^{2kh}e^{kz}) \quad (38)$$

Z function represents the non-oscillatory behavior of the velocity potential, as this potential depends on the wavelength  $\ell$  through  $\lambda = 2\pi/\lambda$ . Put the value  $Z(z)$  of Eq.38 in the below Eq.39

$$\phi(x, z, t) = \sin(kx - \omega t)Z(z) \quad (39)$$

This gives the velocity potential  $\phi$

$$\phi(x, z, t) = \sin(kx - \omega t) \cdot \frac{\omega a}{k(e^{2kh} - 1)} (e^{-kz} + e^{2kh} e^{kz}) \quad (40)$$

This is the final solution equation of boundary value problem. It is a closed form expression. The above equation leads us to the solution of  $\phi$  in terms of angular frequency and wave number. With the help of this equation, velocity potential of liquid at the surface and its movement in forward and reverse direction can be plot.

#### IV. Simulations of Liquid Waves

In this section numerical result obtained from the mathematical model is used to present simulation of dynamic effects of liquid sloshing to demonstrate the effectiveness of angular frequency, wave speed, amplitude, wavelength and height of container carrying the liquid and wave number of sloshing. For this purpose, MATLAB software is used. Sloshing effect is analyzed in MATLAB for three different types of waves to check their behavior and these are

- Dispersive wave
- Moving sinusoidal wave
- Circular wave

##### A. Dispersive Waves Behavior

Figure 4, 5 and 6 are for dispersive wave behavior where wave speed depends on wavelength. To implement this following are the parameters taken for implementation of dispersive wave

- Wave speed= 24m/s
- Amplitude= 3m
- Wavelength= 4m
- The height of container is 10 meter

As we can see in figures that wave number and angular frequency of the waves is changing with sloshing over time.

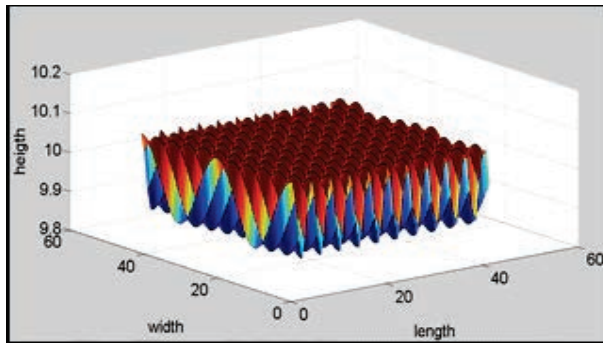


Fig. 4. Motion of Travelling Dispersive Liquid Wave 1st Position

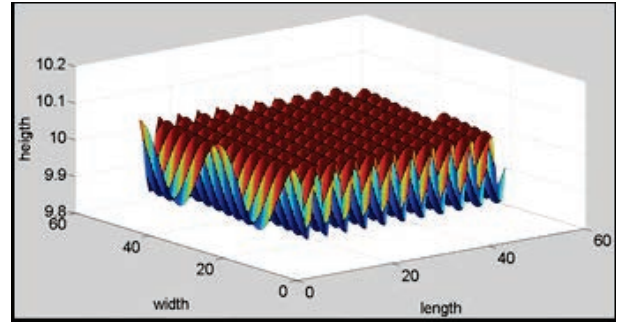


Fig. 5. Motion of Travelling Liquid Wave 2nd Position

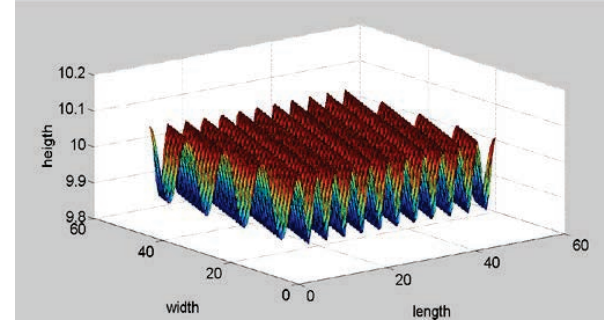


Fig. 6. Motion of Travelling Liquid Wave 3<sup>rd</sup> Position

##### B. Moving Sinusoidal Wave

Figure 7 represents the sloshing effect of liquid in case of environmental disturbances. In this case, liquid surface waves are modeled in a rectangular tank with height, width and length equals to 10 meter and liquid level is 8 meter. It is a travelling sine wave with following parameters

- Frequency= 0.7 Hz
- Wave speed=  $3 \times 10^8$  m/s
- Wavelength= 428571428.5 m

In it angular frequency is continuously changing with sloshing effect.

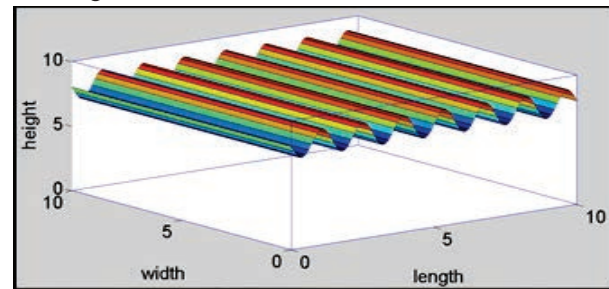


Fig. 7. Moving sinusoid wave

##### C. Circular Wave Behavior

Circular wave propagates with change in amplitude because conservation of energy requires that the amplitude of wave decreases with distance. Wave amplitude gets less as they grow, since their energy is more spread out. Figure 8 shows simulation of circular

wave behavior during sloshing that indicates these waves travelled slowly than the outer waves of shorter wavelengths. There is no change in speed, frequency and wavelength.

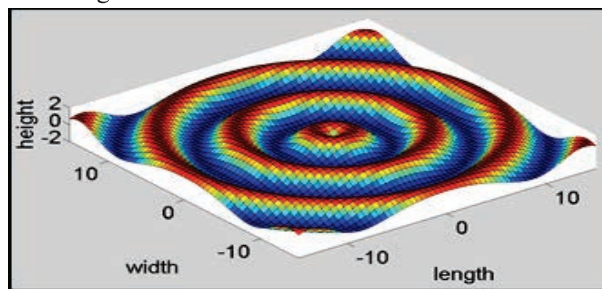


Fig.8. Circular Travelling Wave Behavior

## V. Conclusion

Guided missiles, aerospace vehicles, liquid cargo and maritime applications have conventional control problem that is motion-induced sloshing. Therefore need of accurate evaluation of sloshing is essential. In this research paper, we have formulated a Boundary Value Problem (BVP) of liquid inside a specified container to analyze sloshing effect as a result of external excitations. To reach at final solution, three assumption are considered that Fluid is inviscid, fluid flow is irrotational ( $\nabla \times \mathbf{u} = 0$ ) and  $\nabla \cdot \mathbf{u} = 0$ . Solution is consist of velocity potential ' $\phi$ ', angular frequency ' $\omega$ ' and wave number ' $k$ ' that helps to realize the resonant frequencies in MATLAB for three different waves. The final solution equation can be incorporated with modification of size of vessel by replacing values of  $x$ ,  $z$  and  $h$ . The values of resonant frequencies for our specified conditions are 0.69, 1.93, 2.92, 3.69, 4.31, 4.82 Hz.

Now after finding frequencies, it can be filter out using a low pass filter whose critical frequency should be less than the resonant frequency. The filter will separate out the frequencies causing slosh and variations in the acquired data and give a smooth signal indicating the actual level of the fluid in the container. We can further extent this research by considering many other liquid factors such as viscosity, salinity, density and temperature for more precise measurement of sloshing effect of liquid.

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